

ZERO-DIVISOR GRAPHS OF LOWER DISMANTTABLE LATTICES-II

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ABSTRACT. In this paper, we continue our study of the zero-divisor graphs of lower dismantlable lattices that was started in [20]. The present paper mainly deals with an Isomorphism Problem for the zero-divisor graphs of lattices. In fact, we prove that the zero-divisor graphs of lower dismantlable lattices with the greatest element 1 as join-reducible are isomorphic if and only if the lattices are isomorphic.

Keywords: Dismantlable lattice, adjunct element, zero-divisor graph, rooted tree.

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1. INTRODUCTION

Beck [3] introduced the concept of zero-divisor graph of a commutative ring R with unity as follows. Let G be a simple graph whose vertices are the elements of R and two vertices x and y are adjacent if $xy = 0$. The graph G is known as the *zero-divisor graph* of R . He was mainly interested in the coloring of this graph. This concept is well studied in algebraic structures such as semigroups, rings, lattices, semi-lattices as well as in ordered structures such as posets and qosets; see Alizadeh et al. [1], Anderson et al. [2], Halaš and Jukl [4], Halaš and Länger [5], Joshi [8], Joshi and Khiste [9, 10], Joshi, Waphare and Pourali [12, 13], Lu and Wu [17] and Nimbhorkar et al. [19].

It is easy to observe that if two posets P_1 and P_2 are isomorphic then their zero-divisor graphs $G_{\{0\}}(P_1)$ and $G_{\{0\}}(P_2)$ are isomorphic. But the converse need not be true in general. Hence it is worth to study the following Isomorphism Problem.

Isomorphism Problem: Find a class of posets \mathbb{P} such that $G_{\{0\}}(P_1) \cong G_{\{0\}}(P_2)$ if and only if $P_1 \cong P_2$ for $P_1, P_2 \in \mathbb{P}$.

Joshi and Khiste [9] solved Isomorphism Problem for Boolean posets, which essentially extends the result of LaGrange [16], see also Mohammadian [18]. Recently Joshi, Waphare and Pourali [12] solved this problem for the class of section semi-complemented(SSC) meet semi-lattices.

In the sequel, we obtain a solution for the class of lower dismantlable lattices, a subclass of dismantlable lattices.

In this paper, we continue our study of the zero-divisor graphs of lower dismantlable lattices that was started in [20]. The present paper mainly deals with an Isomorphism Problem of zero-divisor graphs for lower dismantlable lattices. In fact we prove:

Theorem 1.1. *Let \mathcal{L} be the class of lower dismantlable lattices with the greatest element 1 as join-reducible. Then $G_{\{0\}}(L_1) \cong G_{\{0\}}(L_2)$ if and only if $L_1 \cong L_2$ for $L_1, L_2 \in \mathcal{L}$.*

Rival [21] introduced dismantlable lattices to study the combinatorial properties of doubly irreducible elements. By a dismantlable lattice, we mean a lattice which can be completely “dismantled” by removing one element at each stage. Kelly and Rival [15] characterized dismantlable lattices by means of crowns, whereas Thakare, Pawar and Waphare [22] gave a structure theorem for dismantlable lattices using adjunct operation.

Now, we begin with the necessary definitions and terminology.

First, we define the covering relation. We say that ' a is covered by b ', if there is no c such that $a < c < b$ and we denote it by $a \prec b$. Further, a is a lower cover of b and b is an upper cover of a .

Definition 1.2 (Thakare et al. [22]). Let L_1 and L_2 be two disjoint finite lattices and (a, b) is a pair of elements in L_1 such that $a < b$ and $a \not\prec b$. Define the partial order \leq on the set $L = L_1 \cup L_2$ with respect to the pair (a, b) as follows. For $x, y \in L$, we say $x \leq y$ in L if

- either $x, y \in L_1$ and $x \leq y$ in L_1 ; or $x, y \in L_2$ and $x \leq y$ in L_2 ;
- or $x \in L_1, y \in L_2$ and $x \leq a$ in L_1 ; or $x \in L_2, y \in L_1$ and $b \leq y$ in L_1 .

Notice that L is a lattice containing L_1 and L_2 as sublattices. The procedure of obtaining L in this way is called an *adjunct operation of L_2 to L_1* . The pair (a, b) is called an *adjunct pair* and L is an *adjunct* of L_2 to L_1 with respect to the adjunct pair (a, b) and we write $L = L_1]_a^b L_2$ (see Figure 1).

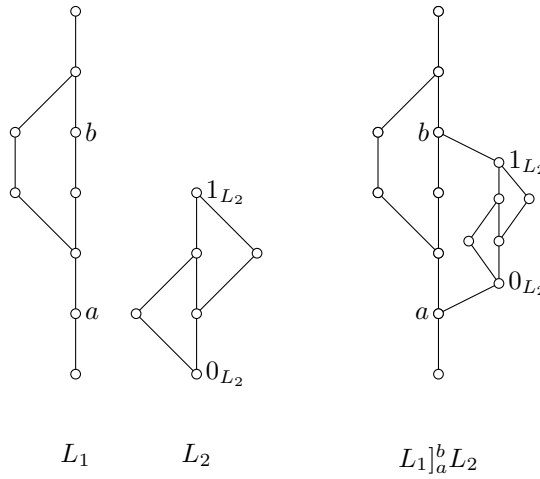


FIGURE 1. Adjunct of two lattices L_1 and L_2

We use the following definition of zero-divisor graph.

Definition 1.3 (Joshi [8]). Let L be a lattice with 0. We associate a simple undirected graph $G_{\{0\}}(L)$ to L . The set of vertices of $G_{\{0\}}(L)$ is $V(G_{\{0\}}(L)) = \{x \in L \setminus \{0\} \mid x \wedge y = 0 \text{ for some } y \in L \setminus \{0\}\}$ and two distinct vertices x, y are adjacent if and only if $x \wedge y = 0$. The graph $G_{\{0\}}(L)$ is called the *zero-divisor graph* of L .

The following result is essentially due to Joshi [8, Theorem 2.4].

Theorem 1.4 (Joshi [8, Theorem 2.4]). *Let L be a lattice with $V(G_{\{0\}}(L)) \neq \emptyset$. Then $G_{\{0\}}(L)$ is connected with $\text{diam}(G_{\{0\}}(L)) \leq 3$.*

An element x in a lattice L is *join-reducible* (meet-reducible) in L if there exist $y, z \in L$ both distinct from x , such that $y \vee z = x$ ($y \wedge z = x$); x is *join-irreducible* (meet-irreducible) if it is not join-reducible (meet-reducible); x is *doubly irreducible* if it is both join-irreducible and meet-irreducible. Therefore, an element x is doubly irreducible in a finite lattice L if and only if x has just one lower cover as well as just one upper cover. The set of all join-irreducible (meet-irreducible) elements of L is denoted by $J(L)$ ($M(L)$). From the definitions of join-irreducibility and meet-irreducibility, it is clear that $0 \in J(L)$ and $1 \in M(L)$. The set of all doubly irreducible elements of L is denoted by $\text{Irr}(L)$ and the set of doubly reducible elements of L is denoted by $\text{Red}(L)$. Thus, if $x \in \text{Red}(L)$

then x is either join-reducible or meet-reducible. A nonzero element p of a lattice L with 0 is an *atom* if $0 \prec p$. The set of atoms in a lattice L is denoted by $At(L)$.

The *cover graph* of a lattice L , denoted by $CG(L)$, is the graph whose vertices are the elements of L and whose edges are the pairs (x, y) with $x, y \in L$ satisfying $x \prec y$ or $y \prec x$. The edge set of $CG(L)$ is denoted by $E(CG(L))$. The *comparability graph* of a lattice L , denoted by $C(L)$, is the graph whose vertices are the elements of L and two vertices x and y are adjacent if and only if x and y are comparable. The complement of the comparability graph, $C(L)^c$, is called the *incomparability graph* of L .

Let x be a vertex of a graph G . The set of neighbors of x in G , denoted by $N(x)$, is given by $\{y \in V(G) \mid x \text{ and } y \text{ are adjacent in } G\}$. The relation defined by $x \sim y$ if and only if $N(x) = N(y)$ is an equivalence relation on $V(G)$. The equivalence class $[x]$ of G is given by $[x] = \{y \in V(G) \mid y \sim x\}$. For undefined notions and terminology from Graph Theory, the reader is referred to West [23].

Definition 1.5 (Rival [21]). A finite lattice L having n elements is *dismantlable*, if there exists a chain $L_1 \subset L_2 \subset \dots \subset L_n (= L)$ of sublattices of L such that $|L_i| = i$, for all i .

Structure Theorem 1.6 (Thakare et al. [22, Theorem 2.2]). A lattice is dismantlable if and only if it is an adjunct of chains.

Thus a dismantlable lattice is of the form $L = (\dots ((C_0]_{a_1}^{b_1} C_1)]_{a_2}^{b_2} C_2) \dots]_{a_r}^{b_r} C_r$, i.e., start with the chain C_0 adjoin C_1 between the pair (a_1, b_1) and so on. If the context is clear, from now onwards, we will simply write $L = C_0]_{a_1}^{b_1} C_1]_{a_2}^{b_2} \dots]_{a_r}^{b_r} C_r$.

Thakare, Pawar and Waphare [22] proved that a dismantlable lattice L need not have a unique adjunct representation but an adjunct pair (a, b) occurs the same number of times in any adjunct representation of L .

Theorem 1.7 (Thakare et al. [22, Theorem 2.7]). A pair (a, b) occurs r times in an adjunct representation of a dismantlable lattice L if and only if there exist exactly $r + 1$ maximal chains C'_0, C'_1, \dots, C'_r in $[a, b]$ such that $x \wedge y = a$ and $x \vee y = b$ for any $x \in C'_i \setminus \{a, b\}$, $y \in C'_j \setminus \{a, b\}$, $i \neq j$.

Now, we recall the definition of a lower dismantlable lattice and its properties from [20].

Definition 1.8. We call a dismantlable lattice L to be a *lower dismantlable*, if it is a chain or every adjunct pair in L is of the form $(0, b)$ for some $b \in L$.

The adjunct representation of a lower dismantlable lattice is of the type $L = C_0]_0^{x_1} C_1]_0^{x_2} \dots]_0^{x_n} C_n$, where C_i 's are chains. We call an element x of a lower dismantlable lattice L an *adjunct element* if $(0, x)$ is an adjunct pair in L .

Remark 1.9. In a lower dismantlable lattice there is no nonzero meet-reducible element.

The following lemma is proved in [20] and gives the properties of lower dismantlable lattices which will be used frequently in sequel.

Lemma 1.10. Let $L = C_0]_0^{x_1} C_1]_0^{x_2} \dots]_0^{x_n} C_n$ be a lower dismantlable lattice, where C_i 's are chains. Then for nonzero elements $a, b \in L$, the following statements are true.

- a) $a \wedge b = 0$ if and only if $a \parallel b$ (where $a \parallel b$ means a and b are incomparable);
- b) Let $a \in C_i$, $b \in C_j$ and $i \neq j$. Then $a \leq b$ if and only if $a = 0$, whenever $i < j$;
 $a \leq b$ if and only if $x_i \leq b$, whenever $j < i$;
- c) If $(0, 1)$ is an adjunct pair (i.e., $x_i = 1$ for some $i \in \{1, 2, \dots, n\}$), then $|V(G_{\{0\}}(L))| = |L| - 2$.

Next result gives the structure of the zero-divisor graph of lower dismantlable lattices.

Theorem 1.11. *A simple undirected graph G is complete k -partite if and only if $G = G_{\{0\}}(L)$ for some lower dismantlable lattice L in which 1 is the only adjunct element.*

Proof. Let G be a complete k -partite graph with V_1, V_2, \dots, V_k as partite sets. Without loss of generality, we can assume that $|V_i| \geq |V_{i+1}|$, for $i = 1, 2, \dots, k-1$. Let C_i be a chain such that $|C_i| = |V_i|$, for $i = 2, 3, \dots, k$ and $|C_1| = |V_1| + 2$. Then form a lattice $L = C_1]_0^1 C_2]_0^1 \dots]_0^1 C_k$. Clearly, L is a lower dismantlable lattice with $(0, 1)$ as the only adjunct pair such that $G_{\{0\}}(L) = G$.

Conversely, suppose that L is a lower dismantlable lattice with 1 as the only adjunct element. Hence $L = C_1]_0^1 C_2]_0^1 \dots]_0^1 C_k$. Then $G_{\{0\}}(L) \neq \emptyset$. In fact, by Lemma 1.10, two vertices are adjacent if and only if they belong to different chains in the adjunct representation of L . Therefore $G_{\{0\}}(L)$ is complete k -partite with $C_1 \setminus \{0, 1\}, C_2, \dots, C_k$ as partite sets. \square

Definition 1.12. Let L be a finite lattice and $x \in L$. We say that x is a *structurally deletable element* of L if $x \in \text{Irr}(L \setminus \{0, 1\})$ and $|E(CG(L))| = |E(CG(L \setminus \{x\}))| + 1$. Delete the structurally deletable element from L and perform the operation of deletion till there does not remain any structurally deletable element. The resultant sublattice of L is called the *basic block associated to L* and it is denoted by $B(L)$.

Illustration 1.13. In Figure 2, a lower dismantlable lattice L and its basic block $B(L)$ from L are depicted. The procedure of obtaining $B(L)$ from L is explained below.

Since $C : a_1 \prec a_2 \prec a_3 \prec a_4$ is a maximal chain of doubly irreducible elements with $0 \prec a_1$, $a_4 \prec x_1$ and $0, x_1 \in \text{Red}(L)$. Then by Definition 1.12, we remove the elements of C except a_1 . Repeat this procedure for the chains $a_9 \prec a_{10} \prec a_{11}$, $a_6 \prec a_7$, $x \prec a_8$. Also remove 1. Lastly, we get the basic block $B(L)$ associated to the lattice L .

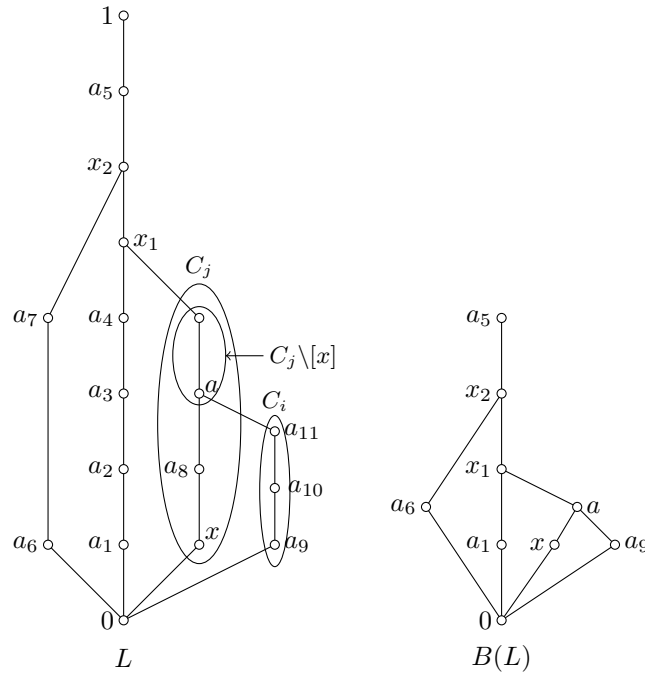


FIGURE 2. A lower dismantlable lattice and its basic block

Observation 1.14. If B is a basic block of a lattice L then an element b is structurally deletable if and only if $x \rightarrow b \rightarrow y$ is the only directed path from x to y in the cover graph $CG(L)$ of L , where $x \prec b \prec y$ in L .

A lattice L with 0 is called *section semi-complemented* (in brief, SSC) if, for any $a, b \in L$ and $a \not\leq b$ there exists $c \in L$ such that $0 < c \leq a$ and $b \wedge c = 0$; see Janowitz [6] (see also Joshi [7]).

Theorem 1.15. Let L be a lower dismantlable lattice such that the greatest element 1 of L is join-reducible. Then the following statements are equivalent.

- (a) The basic block of L is L itself.
- (b) L is SSC.
- (c) Every equivalence class of the zero-divisor graph $G_{\{0\}}(L)$ is singleton.

Proof. (a) \Rightarrow (b) Suppose that L is a basic block of L itself. On the contrary, suppose L is not SSC. Hence there exists a pair of elements $b \not\leq a$ with $a \wedge x \neq 0$ for every nonzero $x \leq b$. Then every nonzero $x \leq b$ is comparable with a . We claim that b is a join-irreducible element. Suppose b is not join-irreducible, then $b = x \vee y$ for $x, y \prec b$. Clearly $a \not\leq x, y$. Otherwise, since L is lower dismantlable, $x \parallel y$. We have by Lemma 1.10, $x \wedge y = 0$. Hence $a = 0$, a contradiction to $a \wedge x \neq 0$ for any $x \leq b$. But then the only possibility is $x, y \leq a$ which yields $b = x \vee y \leq a$, again a contradiction. Thus b is join-irreducible. Since L is a lower dismantlable lattice, we get $L = C_0]_0^{a_1} C_1]_0^{a_2} \cdots]_0^{a_n} C_n$, where each C_i is a chain. By Remark 1.9, every nonzero element of L is meet-irreducible. In particular, b is meet irreducible, hence b is doubly irreducible. Let w_1 and w_2 be the unique upper and lower covers of b respectively. Then $w_1 \rightarrow b \rightarrow w_2$ is the unique directed path from w_1 to w_2 in the cover graph of L , i.e., $CG(L)$. Hence the element b is structurally deletable and by deleting b , there is an edge joining w_1 to w_2 , a contradiction to the fact that L is a basic block of L itself. Hence L is SSC.

(b) \Rightarrow (a) Suppose L is SSC. If L is not a basic block of L itself, then there exists a doubly irreducible element, say b , which is structurally deletable. We claim that b is an atom. Suppose on the contrary that there exists nonzero element $x \prec b$. Then by the definition of SSC, there exists nonzero $c \leq b$ such that $x \wedge c = 0$. Since $x \prec b$ and $c \leq b$, we get $x \vee c = b$, a contradiction to the choice of b . Hence b is an atom in L . Let w be the unique upper cover of b . Then $0 \rightarrow b \rightarrow w$ is a directed path from 0 to w in $CG(L)$. If w is join-reducible, then there is another path $0 \rightarrow c \rightarrow w$ in $CG(L)$, hence b can not be structurally deletable, a contradiction. If w is join-irreducible, then $0 \rightarrow b \rightarrow w$ is the only directed path from 0 to w in $CG(L)$. Thus $b \leq w$ and there is no element $y \leq w$ such that $b \wedge y = 0$, a contradiction, as L is SSC. Hence L is a basic block of L itself.

(c) \Leftrightarrow (b) follows from Joshi et al. [14, Lemma 9]. \square

Joshi et al. [14] proved that Isomorphism Problem is true for the class of SSC meet semi-lattices.

Theorem 1.16 (Joshi et al. [14]). Let \mathcal{L} denotes the class of SSC meet semi-lattices. Then $L_1 \cong L_2$ if and only if $G_{\{0\}}(L_1) \cong G_{\{0\}}(L_2)$ for $L_1, L_2 \in \mathcal{L}$.

Remark 1.17. In view of Theorem 1.15 and Theorem 1.16, it is clear that Isomorphism Problem is true for the class of lower dismantlable lattices which are basic block of itself.

In the next section, we prove that Isomorphism Problem is true for the larger class of lattices, namely, the class of lower dismantlable lattices.

2. MAIN RESULTS

Let T be a rooted tree with the root R having at least two branches. Let $G(T)$ be the *non-ancestor graph* of T with vertex set $V(G(T)) = T \setminus \{R\}$ and two vertices are adjacent in $G(T)$ if and only if no one is an ancestor of the other. Denote the class of non-ancestor graphs of rooted trees by \mathcal{G}_T .

In [20], the following result is proved for the zero-divisor graphs of lower dismantlable lattices.

Theorem 2.1 (Patil et al. [20]). *For a simple undirected graph G , the following statements are equivalent.*

- (a) $G \in \mathcal{G}_T$, the class of non-ancestor graphs of rooted trees.
- (b) $G = G_{\{0\}}(L)$ for some lower dismantlable lattice L with the greatest element 1 as a join-reducible element.
- (c) G is the incomparability graph of $(L \setminus \{0, 1\}, \leq)$ for some lower dismantlable lattice L with the greatest element 1 as a join-reducible element.

Remark 2.2. From Theorem 2.1, it is clear that there is a one-to-one correspondence between the lower dismantlable lattices and rooted trees. In fact, a lattice L is a lower dismantlable lattice if and only if $L \setminus \{0\}$ is a rooted tree with the root 1. On the other hand, given a rooted tree T with the root R , we join an element say 0 to all the pendent vertices of T and get a cover graph of a lower dismantlable lattice L in which R is the greatest element and 0 is the smallest element of L . We call T as the *corresponding rooted tree* of L . Hence in view of Theorem 2.1, it is clear that $G_{\{0\}}(L) = G(T)$ for a lower dismantlable lattice L and its corresponding rooted tree T . Therefore the equivalence classes of $G(T)$ are same as the equivalence classes of $G_{\{0\}}(L)$.

Note: In a lower dismantlable lattice which is not a chain, every adjunct element contains at least two atoms.

In the following construction, we give an algorithm to determine all equivalence classes of $G_{\{0\}}(L)$, where L is a lower dismantlable lattice.

Construction 2.3. Let T be a rooted tree and $G(T)$ be the non-ancestor graph of T . A vertex v of T is a *node* if the total degree $\deg(v) > 2$ in T . If L is a lower dismantlable lattice with the corresponding rooted tree T , then $a (\neq 1)$ is an adjunct element in L if and only if a is a node in T .

Let v be a node of T such that no successor of v is a node of $G(T)$. Then each branch with a successor of v , i.e., a directed path in T of which every element is a successor of v , is an equivalence class in $G(T)$ under the relation \sim (i.e. having same neighbors). Delete all such branches and look at the resultant rooted tree T' . Repeat this process in T' and so on, we get all the equivalence classes of $G(T)$.

We illustrate this procedure with an example.

Example 2.4. In Figure 3, a lower dismantlable lattice L , its corresponding rooted tree T and its zero-divisor graph $G_{\{0\}}(L)$ are depicted. In the corresponding rooted tree T , a_5, a_6 are nodes having no successor as a node, whereas a_8 is a node having a_5 and a_6 as successor nodes. Hence delete vertices a_1, a_2, a_3, a_4, a_7 . This gives the equivalence classes $\{a_1, a_7\}, \{a_2\}, \{a_3\}$, and $\{a_4\}$. Note that these equivalence classes do not contain an adjunct element. Now, the resultant rooted tree with the root 1 and a_8 is a node without successor node. Therefore $\{a_5\}$ and $\{a_6\}$ are the equivalence classes of $G(T)$ which contain an adjunct element of L . In the last stage we get $\{a_8\}$ as an equivalence class. Thus in this way, we get all the equivalence classes of $G(T)$.

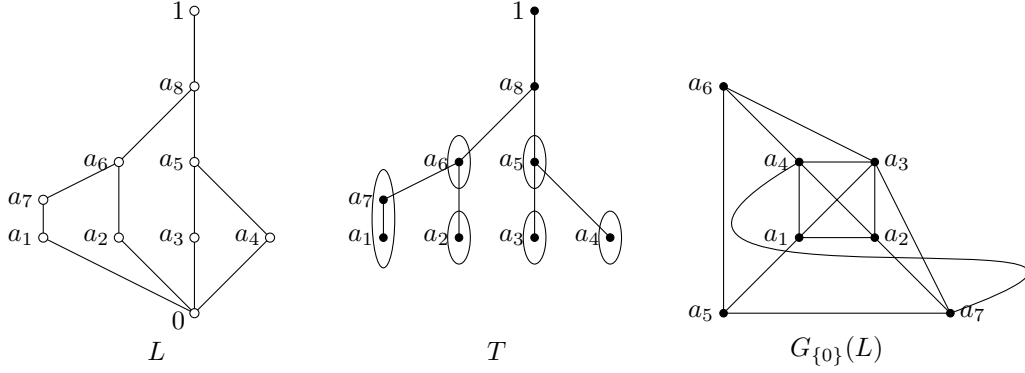


FIGURE 3. A lower dismantlable lattice with its corresponding rooted tree and zero-divisor graph

The following lemma gives more about equivalence classes of $G_{\{0\}}(L)$.

Lemma 2.5. *Let L be a lower dismantlable lattice and $G_{\{0\}}(L)$ be its zero-divisor graph. Then the following statements are true.*

- (a) *Every equivalence class of $G_{\{0\}}(L)$ forms a chain in L . Moreover, every equivalence class contains at most one adjunct element.*
- (b) *If a is an adjunct element in the equivalence class $[x]$ of $G_{\{0\}}(L)$, then $a \leq x$.*
- (c) *An equivalence class $[x]$ of $G_{\{0\}}(L)$ contains an adjunct element of L if and only if $[x]$ of $G(T)(= G_{\{0\}}(L))$ does not contain a pendent vertex of the corresponding rooted tree T (mentioned in Remark 2.2).*
- (d) *The branches that we get by the procedure explained in Example 2.4 are precisely the equivalence classes of $G_{\{0\}}(L)$.*

Proof. (a) Let $[x] = \{y \in V(G_{\{0\}}(L)) \mid N(x) = N(y)\}$. Let $y_1, y_2 \in [x]$. If $y_1 \parallel y_2$, then $y_1 \wedge y_2 = 0$, by Lemma 1.10. Hence $y_1 \in N(y_2) = N(y_1)$, a contradiction. Hence y_1 and y_2 are comparable. Therefore $[x]$ is a chain. Next, suppose $y, z \in [x]$ be two adjunct elements. Without loss of generality, let $y < z$. Since z is an adjunct element, there exists a nonzero $w \leq z$ such that $y \wedge w = 0$. Hence $w \in N(y) = N(z)$ which yields $w = w \wedge z = 0$, a contradiction. Therefore every equivalence class contains at the most one adjunct element.

(b) As $a \sim x$, we get a and x are comparable. If $x < a$, then there exists a nonzero $b < a$ such that $x \wedge b = 0$ (as a is an adjunct element). Hence $b \in N(x) = N(a)$ which gives $b = a \wedge b = 0$, a contradiction. Therefore $a \leq x$.

(c) It is easy to observe that an element y is an atom in L if and only if y is a pendent vertex of the corresponding rooted tree T of L . Suppose that a be an adjunct element in $[x]$. Hence $a \leq x$. On the contrary, assume that $p \in [x]$ be an atom. Then $p \leq a$. Since a is an adjunct element, there exists an atom $q (\neq p) \leq a$. Hence $p \wedge q = 0$, which gives $q \in N(p) = N(a)$. This yields $q = q \wedge a = 0$, a contradiction. Therefore $[x]$ contains no atom. Conversely, suppose that $[x]$ of $G(T)$ contains no atom of L . Let p be an atom such that $p \leq x$. This gives $N(x) \subsetneq N(p)$. Since $[x]$ contains no atom, there exists $y \in N(p)$ such that $y \wedge x \neq 0$. Then x and y are comparable, by Lemma 1.10. Since $p \leq x$, the case $x \leq y$ is impossible. Hence $y < x$. Then $p \vee y \leq x$. If $p \vee y \in [x]$ then we are through, as $p \vee y$ is an adjunct element. If not, i.e., $p \vee y \notin [x]$ then again $N(x) \subsetneq N(p \vee y)$. So there exists $c_1 \in N(p \vee y)$ such that $c_1 \notin N(x)$. This gives x and c_1 are comparable, by Lemma 1.10 (a). Using the above arguments, we get $c_1 \leq x$ and $N(x) \subseteq N(p \vee y \vee c_1)$. Continuing in this way we get an element say c_n (as L is finite) such that $N(x) = N(p \vee y \vee_{i=1}^n c_i)$. Then $p \vee y \vee_{i=1}^n c_i$ is an adjunct

element such that $y \vee z \vee \bigvee_{i=1}^n c_i \in [x]$.

(d) Follows from the fact that the equivalence classes of $G(T)$ are same as the equivalence classes of $G_{\{0\}}(L)$. \square

Corollary 2.6. *An equivalence class $[x]$ of $G_{\{0\}}(L)$ for a lower dismantlable lattice L contains an adjunct element if and only if there is a pair of vertices $y, z \in V(G_{\{0\}}(L))$ such that y is adjacent to z and x is not adjacent to any of y and z .*

Proof. Suppose a is an adjunct element in $[x]$. Hence $a \leq x$. Since a is an adjunct element, there exist two atoms $p_1, p_2 \leq a$. Then p_1 and p_2 are the required elements. Conversely, suppose that there is a pair of vertices $y, z \in V(G_{\{0\}}(L))$ such that y is adjacent to z and x is not adjacent to any of y and z . Let p, q be atoms such that $p \in [x]$ and $q \leq y$. Since x and y are non-adjacent, they are comparable in L . The case $x \leq y$ is impossible, since y and z are adjacent but x and z are non-adjacent. Hence $y < x$, which further gives $q \leq x$. But $p \wedge q = 0$ and $p \in [x]$ gives $q \in N(p) = N(x)$ which yields $q \wedge x = 0$, a contradiction. Hence $[x]$ does not contain an atom. By Lemma 2.5, $[x]$ does not contain an adjunct element. \square

We need the following concept of ordinal sum of two posets.

Definition 2.7. Let P and Q be disjoint posets. Let $P \cup Q$ be the union with the inherited order on P and Q such that $p < q$ for all $p \in P$ and $q \in Q$. Then it forms a poset called the *ordinal sum* of P and Q denoted by $P \oplus Q$.

Lemma 2.8. *If an equivalence class $[x]$ does not contain an adjunct element then the set $A_x = \{b \in G_{\{0\}}(L) \mid b \text{ is an adjunct element of } L \text{ not adjacent to } x\}$ is either empty or forms a chain in L . In this case, $L = L_1 \oplus_a C$, where L_1 is a lower dismantlable lattice and C is a chain. Moreover, $a = 1$ if and only if $A_x = \emptyset$, and if $a \neq 1$, then a is the smallest element of A_x .*

Proof. Suppose $[x]$ does not contain any adjunct element. Let $L = C_0]_0^{x_1} C_1]_0^{x_2} C_2]_0^{x_3} \cdots]_0^{x_n} C_n$ and $x \in C_j$. Consider the set $A_x = \{b \in L \mid b \text{ is an adjunct element not adjacent to } x\}$

If $A_x = \emptyset$, then C_j must be joined at $(0, 1)$ and all the elements of C_j have same neighbors, i.e., $[x] = C_j$. Then $L = L_1]_0^1 C_j$, where $L_1 = C_0]_0^{x_1} C_1]_0^{x_2} \cdots]_0^{x_{j-1}} C_{j-1}]_0^{x_{j+1}} C_{j+1}]_0^{x_{j+2}} \cdots]_0^{x_n} C_n$.

Let $A_x \neq \emptyset$. Suppose $b \in A_x$. Then x is not adjacent to b . Hence x and b are comparable. Suppose $b \leq x$. Since b is an adjunct element, there exists a pair $y, z \leq b$ such that $y \wedge z = 0$ and x is not adjacent to any of y and z . By Corollary 2.6, $[x]$ contains an adjunct element, a contradiction. Therefore $x \leq b$.

Now, we prove that the elements of A_x forms a chain. Let $b_1, b_2 \in A_x$ such that $b_1 \parallel b_2$. Then $b_1 \wedge b_2 = 0$. As $b_1, b_2 \in A$, we have $x \leq b_1$ and $x \leq b_2$. Therefore $x \wedge b_1 = 0$, i.e., x and b_1 are adjacent, a contradiction to the choice of b_1 . Therefore the elements of A_x forms a chain.

Let a be the smallest element of A_x . We claim that $[x] \subseteq C_j$. Let $y \sim x$, i.e., $y \wedge z = 0$ if and only if $x \wedge z = 0$. As $x \wedge y \neq 0$, they are comparable. If $y \leq x$, then it gives $y \in C_j$, otherwise $[x]$ contains an adjunct element, a contradiction. Suppose $x \leq y$ and $y \in C_i$ for $i \neq j$. By Lemma 1.10 (b), we get $y \geq x_j$, where x_j is an adjunct element. Therefore there exist two elements $y_1, y_2 \leq x_j$ such that $y_1 \wedge y_2 = 0$. As $x_j \leq y$, we have y_1, y_2 are adjacent and y is not adjacent to any of them. By Corollary 2.6, $[y] = [x]$ (as $y \sim x$) contains an adjunct element, a contradiction to the assumption. Hence $y \in C_j$, i.e., $[x] \subseteq C_j$.

Next, we claim that if $z \in C_j$ such that $z < a$, then $z \sim x$. Let $z \in C_j$ such that $z < a$. Since $x, z \in C_j$, they are comparable.

If $x \leq z$, then $N(z) \subseteq N(x)$. If there exists $y \in N(x)$ but $y \notin N(z)$, then y and z are comparable. If $z \leq y$, then $z \wedge x = 0$, a contradiction. Hence $y \leq z$. Thus $x, y \leq z$ such that $x \wedge y = 0$. Hence by Lemma 2.5 (b), there exists an adjunct element, say $c \in [z]$, such that $x \leq c \leq z$, a contradiction to the minimality of a . Hence $x \leq z$ gives $N(x) = N(z)$.

Suppose that $z \leq x$. Then $N(x) \subseteq N(z)$. If $y \in N(z)$ such that $y \notin N(x)$, then y and x are comparable. If $x \leq y$, then $x \wedge z = 0$, a contradiction. Hence $y \leq x$. This together with $z \leq x$, $y \in N(z)$ and by Corollary 2.6, we have $[x]$ contains an adjunct element, again a contradiction. Hence $N(z) = N(x)$, *i.e.*, $x \sim z$.

Further, suppose that an adjunct element x_i is non-adjacent to x . Hence x_i and x are comparable. If $x_i \leq x$, then by Corollary 2.6, $[x]$ contains an adjunct element, a contradiction. Hence $x < x_i$. In particular, we have $x < a$.

Now, we prove that $L = L_1]_0^a C$, where L_1 is a lower dismantlable lattice and C is a chain. First, assume that $x \notin C_0$, *i.e.*, $x \in C_j$ for $j \neq 0$. Since $x < a$, we have the following two cases.

Case (1): If $a \notin C_j$. Since $x < a$ and $a \notin C_j$, by Lemma 1.10(b), we get $x_j \leq a$. By the minimality of a , we have $a = x_j$ and no element of C_j is an adjunct element. In this case $[x] = C_j$, as all elements of C_j have same neighbors. Define $L' = L_1]_0^a C_j$ with the induced partial order of L , where $L_1 = C_0]_0^{x_1} C_1]_0^{x_2} C_2]_0^{x_3} \dots]_0^{x_{j-1}} C_{j-1}]_0^{x_{j+1}} C_{j+1}]_0^{x_{j+2}} \dots]_0^{x_n} C_n$.

We claim that $L' = L$. Clearly, $|L| = |L'|$ (as we are playing with the same elements). Let $y_1 \leq y_2$ in L . If $y_1, y_2 \notin C_j$ or $y_1, y_2 \in C_j$, then $y_1 \leq y_2$ in L' . Note that $y_2 \in C_j$ implies $y_1 \in C_j$, since no element of $C_j = [x]$ is an adjunct element. Suppose $y_1 \in C_j$ and $y_2 \in C_i$ for $i \neq j$ in L . Since $y_1 \leq y_2$ and $y_1 \notin C_i$, we get $x_j \leq y_2$ in L . Hence $y_1 \leq x_j \leq y_2$ in L' . The converse follows similarly. Therefore $L = L' = L_1]_0^a C_j$.

Case (2): Let $a \in C_j$. Then a and x are on the same chain C_j . Since a is an adjunct element; we have $a = x_i$, for some i . Hence C_i is a chain joined at a .

Define $L_1 = C_0]_0^{x_1} C_1]_0^{x_2} \dots]_0^{x_{j-1}} C_{j-1}]_0^{x_j} C'_j]_0^{x_{j+1}} C_{j+1}]_0^{x_{j+2}} \dots]_0^{x_{i-1}} C_{i-1}]_0^{x_{i+1}} C_{i+1} \dots]_0^{x_n} C_n$, where $C'_j = C_i \oplus (C_j \setminus [x])$ (see Figure 2). Since $a \in C_j$ and $a \notin [x]$, $C_j \setminus [x] \neq \emptyset$. Let $L' = L_1]_0^{a=x_i} C$, where $C = [x]$ is a chain (by Corollary 2.6) under the induced partial order of L . We claim that $L = L'$. Observe that $|L| = |L'|$ (as we are playing with the same elements).

Suppose $y_1 \leq y_2$ in L . If they belong to the same chain, say C_k ($k \neq j$) in L , then $y_1 \leq y_2$ in L' also.

Suppose $y_1, y_2 \in C_j$. If $y_1, y_2 \sim x$, then $y_1, y_2 \in C$ in L' implies $y_1 \leq y_2$ in L' . If $y_1, y_2 \not\sim x$, then $y_1, y_2 \in C'_j$ in L_1 implies $y_1 \leq y_2$ in L' . Suppose $y_1 \sim x$ and $y_2 \not\sim x$, then $y_1 \in C$ and $y_2 \in C'_j$ in L' . As $y_1 \sim x$, $y_2 \in C'_j$ and a is an adjunct element, we have $y_1 < a$ and $y_2 \geq a$, which gives $y_1 \leq y_2$ in L' .

Now, suppose y_1 and y_2 are on different chains, say $y_1 \in C_p$ and $y_2 \in C_k$ with $p \neq k$. As $y_1 \leq y_2$ in L , we get $y_2 \geq x_p$ (as C_p is joined at x_p) hence $k < p$. Also $y_1 \leq x_p$, hence $y_1 \leq x_p \leq y_2$ in L' .

Next, suppose $y_1 \leq y_2$ in L' . If they belong to the same chain except C'_j , then $y_1 \leq y_2$ in L also. Suppose $y_1, y_2 \in C'_j$. If $y_1, y_2 \in C_j \setminus [x]$ or $y_1, y_2 \in C_i$, then $y_1 \leq y_2$ in L . If $y_2 \in C_i$, then $y_1 \in C_i$, as $y_1 \leq y_2$. Let $y_2 \in C_j \setminus [x]$ and $y_1 \in C_i$. Then $y_1 \leq a \leq y_2$ in L . Suppose y_1 and y_2 are on different chains, say $y_1 \in C_p$ and $y_2 \in C_k$ with $p \neq k$ in L' . As $y_1 \leq y_2$, we have $k < p$ with $y_2 \geq x_p$ and $x_p \geq y_1$ *i.e.*, $y_1 \leq x_p \leq y_2$ in L .

Therefore $L = L'$.

If $x \in C_0$, it gives $a \in C_0$. Then x and a are on the same chain C_0 , hence by Case (2) above, we have $L = L_1]_0^a C$, where $C = [x]$. \square

Theorem 2.9. *Let L_1, L_2 be lower dismantlable lattices with 1 as an adjunct element such that $G_{\{0\}}(L_1) \cong G_{\{0\}}(L_2)$. Then there is a graph isomorphism $\phi : G_{\{0\}}(L_1) \rightarrow G_{\{0\}}(L_2)$ such that a is an adjunct element of L_1 if and only if $\phi(a)$ is an adjunct element of L_2 .*

Proof. Let $f : G_{\{0\}}(L_1) \rightarrow G_{\{0\}}(L_2)$ be a graph isomorphism, where L_1 and L_2 be lower dismantlable lattices. Since 1 is an adjunct element, we have $V(G_{\{0\}}(L_i)) = L_i \setminus \{0, 1\}$, for $i = 1, 2$. First, we prove that $f([x]) = [f(x)]$ for all $x \in V(G_{\{0\}}(L_1))$, where $f([x]) = \{f(a) \mid a \in [x]\}$ and $[x] = \{y \in L_1 \mid N(x) = N(y)\}$.

Let $t \in f([x])$. Then $t = f(a)$ for some $a \in [x]$. Therefore $N(a) = N(x)$ implies $N(f(a)) = N(f(x))$ by the graph isomorphism. Hence $t \in [f(x)]$. Thus $f([x]) \subseteq [f(x)]$. Let $f(a) \in [f(x)]$. Hence $N(f(a)) = N(f(x))$ implies $N(a) = N(x)$ by the graph isomorphism. Therefore $a \in [x]$ which yields $f(a) \in f([x])$. Hence $[f(x)] \subseteq f([x])$. Thus $[f(x)] = f([x])$.

It is clear that $y \sim x$ if and only if $f(y) \sim f(x)$. This gives $|[x]| = |[f(x)]|$.

Now, we claim that $[x]$ contains an adjunct element if and only if $[f(x)]$ contains an adjunct element. Suppose $[x]$ contains an adjunct element of L_1 . Then by Corollary 2.6, there exist elements $y, z \in L_1$ such that x is not adjacent to any of y and z with $y \wedge z = 0$. Hence $f(x)$ is not adjacent to any of $f(y)$ and $f(z)$ with $f(y) \wedge f(z) = 0$. Therefore $[f(x)]$ contains an adjunct element of L_2 . Converse follows on the similar lines.

Define $A_f = \{x \mid x \text{ is an adjunct element of } L_1 \text{ and } f(x) \text{ is not an adjunct element of } L_2\}$. We prove the result by the induction on $|A_f|$.

If $A_f = \emptyset$, then f is the required isomorphism.

Let $A_f \neq \emptyset$ and assume the result is true for all lower dismantlable lattices with $|A_f| < k$. Now, suppose $|A_f| = k$. Let $A_f = \{x_1, x_2, \dots, x_k\}$. Since x_i is an adjunct element, $[x_i]$ contains an adjunct element. Hence $[f(x_i)]$ contains an adjunct element, but $f(x_i)$ is not an adjunct element, as $x_i \in A_f$. So $|[x_i]| = |[f(x_i)]| > 1$ for $i = 1, 2, \dots, k$. Let $[x_1] = \{x_{11}, x_{12}, \dots, x_{1m}\}$ with $x_1 = x_{11}$. Then $[f(x_1)] = \{f(x_{11}), f(x_{12}), \dots, f(x_{1m})\}$. Without loss of generality, suppose $f(x_{12})$ be an adjunct element of L_2 .

Next define $\phi_1 : G_{\{0\}}(L_1) \rightarrow G_{\{0\}}(L_2)$ by $\phi_1(y) = f(y)$, for all $y \in L_1 \setminus \{x_{11}, x_{12}\}$, $\phi_1(x_1) = f(x_{12})$ and $\phi_1(x_{12}) = f(x_1)$. Then ϕ_1 is bijective. Let x and y be adjacent in $G_{\{0\}}(L_1)$. Then $|\{x, y\} \cap \{x_{11}, x_{12}\}| < 2$. Note that $f(x_1) = f(x_{11})$ is not an adjunct element, as x_{11} and x_{12} are non-adjacent.

If $\{x, y\} \cap \{x_{11}, x_{12}\} = \emptyset$, then $x, y \in L_1 \setminus \{x_{11}, x_{12}\}$. Hence $\phi_1(x)$ and $\phi_1(y)$ are adjacent in $G_{\{0\}}(L_2)$ by the definition of ϕ_1 . Suppose $|\{x, y\} \cap \{x_{11}, x_{12}\}| = 1$. Without loss of generality, suppose that $x = x_{11}$. As $x_{11} \sim x_{12}$, we have x_{12} and y are adjacent in $G_{\{0\}}(L_1)$. Therefore $f(x_{12})$ and $f(y)$, i.e., $\phi_1(x)$ and $\phi_1(y)$ are adjacent in $G_{\{0\}}(L_2)$. Also using the above arguments for ϕ_1^{-1} , we have x and y are adjacent in $G_{\{0\}}(L_1)$ whenever $\phi_1(x)$ and $\phi_1(y)$ are adjacent in $G_{\{0\}}(L_2)$. Hence ϕ_1 is a graph isomorphism.

Let $a \in A_{\phi_1}$. Then a is an adjunct element of L_1 and $\phi_1(a)$ is not an adjunct element of L_2 . We claim that $a \notin \{x_{11}, x_{12}\}$. On the contrary, assume that $a \in \{x_{11}, x_{12}\}$. If $a = x_{11} = x_1$, then $\phi_1(a) = \phi_1(x_{11}) = f(x_{12})$ is an adjunct element of L_2 , a contradiction. Also $a = x_{12}$ impossible because a is an adjunct element of L_1 but x_{12} is not, since each equivalence class contains at the most one adjunct element and $x_1 = x_{11}$ is an adjunct element in $[x_1]$ with $x_{12} \in [x_1]$. Therefore $a \notin \{x_{11}, x_{12}\}$. This gives $f(a) = \phi_1(a)$ is not an adjunct element of L_2 , hence $a \in A_f$. Thus $A_{\phi_1} \subseteq A_f$. But $x_1 \in A_f$ with $x_1 \notin A_{\phi_1}$. This gives $A_{\phi_1} \subsetneq A_f$. Therefore $|A_{\phi_1}| < |A_f|$. Hence by induction, there is an isomorphism $\phi : G_{\{0\}}(L_1) \rightarrow G_{\{0\}}(L_2)$ such that a is an adjunct element of L_1 if and only if $\phi(a)$ is an adjunct element of L_2 . Hence the result. \square

Theorem 2.10. *Let L_1 and L_2 be lower dismantlable lattices with 1 as adjunct element. If $\phi : G_{\{0\}}(L_1) \rightarrow G_{\{0\}}(L_2)$ is a graph isomorphism such that a is an adjunct element in L_1 if and only if $\phi(a)$ is an adjunct element in L_2 . Then there exists an isomorphism $\psi : L_1 \rightarrow L_2$ such that $\psi|_X \equiv \phi|_X$, where X is the set of adjunct elements of L_1 different from 1. Moreover, for any equivalence class $[x]$ in $G_{\{0\}}(L_1)$, we have $\psi([x]) = \phi([x])$.*

Proof. We use the induction on the number of vertices. By Theorem 1.4, $G_{\{0\}}(L_i)$ are connected, for $i = 1, 2$. We know that if there are only two vertices then the graphs $G_{\{0\}}(L_i)$, $i = 1, 2$ are isomorphic to K_2 and therefore the lattices are isomorphic to the power set of two elements. In this case, 1 is the only adjunct element, hence $X = \emptyset$.

Now, suppose $G_{\{0\}}(L_1) \cong G_{\{0\}}(L_2)$ with $|G_{\{0\}}(L_i)| > 2$, for $i = 1, 2$. Suppose L_1 and L_2 satisfy the hypothesis. Select $x \in L_1$ such that the equivalence class $[x]$ in $G_{\{0\}}(L_1)$ does not contain any adjunct element of L_1 . Note that, by Lemma 2.5, such an equivalence class $[x]$ exists. Then by the hypothesis $[\phi(x)]$ also does not contain any adjunct element of L_2 . By Lemma 2.8, we can write $L_1 = L_1]_0^a C$ and $L_2 = L_2]_0^{a'} C'$, where $C = [x]$ and $C' = [\phi(x)]$ and either a, a' both are corresponding 1's of L_1 and L_2 respectively or the smallest elements of corresponding set of all adjunct elements which are comparable with $x, \phi(x)$ respectively.

We claim that $\phi(a) = a'$. As a and x are non-adjacent in $G_{\{0\}}(L_1)$, we have $\phi(a)$ and $\phi(x)$ are non-adjacent in $G_{\{0\}}(L_2)$. Hence they are comparable in L_2 with $\phi(a)$ as an adjunct element in L_2 . But a' is the smallest adjunct element which is not adjacent to $\phi(x)$, hence $a' \leq \phi(a)$. Let $b \in L_1$ such that $\phi(b) = a'$. Then b is an adjunct element in L_1 since $\phi(a)$ is an adjunct element and if $a' < \phi(a)$, then there exists an element $\phi(c) < \phi(a)$ such that $a' \wedge \phi(c) = 0$, i.e., $\phi(b) \wedge \phi(c) = 0$. By the graph isomorphism, $b \wedge c = 0$ in L_1 . Also, both b and c are comparable with a . The only possibility is $b, c < a$. If b is adjacent to x in $G_{\{0\}}(L_1)$, then $a' = \phi(b)$ is adjacent to $\phi(x)$ in $G_{\{0\}}(L_2)$, a contradiction to the fact that $\phi(x)$ and $\phi(b)$ are comparable, as $\phi(x)$ is on the chain C' and C' is joined at $a' = \phi(b)$. Hence b is not adjacent to x in $G_{\{0\}}(L_1)$ and b is an adjunct element with $b < a$, a contradiction to the smallestness of a . Hence $a' = \phi(a)$.

Among all such equivalence classes select one $[x]$ for which corresponding element a is minimal among such adjunct elements (minimal in the sense that, if b another adjunct element, then either $a \parallel b$ or $a \leq b$). Then $[\phi(x)]$ is an equivalence class in $G_{\{0\}}(L_2)$ such that a' is minimal among such adjunct elements of L_2 . Next, we consider the following cases for a .

Case (1): Suppose $a = 1$. In this case there is no adjunct pair other than $(0, 1)$, hence $X = \emptyset$ and the result follows by Theorem 1.11.

Case (2): $a \neq 1$. Then we have $a' \neq 1$, otherwise 1 is the only adjunct element in L_2 (by the minimality of a'). By Theorem 1.11, we get $G_{\{0\}}(L_2)$ is complete bipartite, so is $G_{\{0\}}(L_1)$ and again by Theorem 1.11, we get 1 is the only adjunct element in L_1 , a contradiction to the fact that $a \neq 1$. Thus we must have $a' = 1$. Consequently, in the lower dismantlable lattices L'_1 and L'_2 , the corresponding greatest elements 1 are join-reducible. Note that $|[x]| = |[\phi(x)]|$, i.e., $|C| = |C'|$ follows as in the proof of Theorem 2.9. Also $G_{\{0\}}(L'_1) = G_{\{0\}}(L_1) \setminus [x] \cong G_{\{0\}}(L_2) \setminus [\phi(x)] = G_{\{0\}}(L'_2)$, under the map $\phi|_{L'_1}$. Hence by the induction hypothesis, there exists an isomorphism $\psi : L'_1 \rightarrow L'_2$ such that $\psi|_{X_1} \equiv \phi|_{X_1}$, where X_1 is the set of adjunct elements of L'_1 different from 1 and for any equivalence class $[y]$ in $G_{\{0\}}(L'_1)$, $\psi([y]) = \phi([y])$.

Suppose a is an adjunct element in L'_1 . Then $\psi(a)$ is an adjunct element in L'_2 with $\psi(a) = \phi(a) = a'$. Since C and C' are chains of same length both without containing an adjunct element, hence we can extend ψ to $L_1 = L'_1]_0^a C$ which gives $L_1 \cong L_2$ and $\psi|_X \equiv \phi|_X$, where X is the set of

adjunct elements of L_1 different from 1, and in this case $X = X_1$. Also, for any equivalence class $[x]$ in $G_{\{0\}}(L_1)$, we have $\psi([x]) = \phi([x])$.

Now, suppose a is not an adjunct element in L'_1 . We claim that $[a]$ in $G_{\{0\}}(L'_1)$ does not contain any adjunct element. If $b \in [a]$ be an adjunct element in $G_{\{0\}}(L'_1)$, then by Lemma 2.5 (a), we get $b < a$ in L'_1 and hence in L_1 . Let C_a be a chain in the adjunct representation of L_1 such that $a \in C_a$ and y is an atom of L_1 on the chain C_a . As $a, b \in L'_1$ and there is only one chain C joined at a in L_1 , we have $b \in C_a$. Also $[y]$ in $G_{\{0\}}(L_1)$ does not contain any adjunct element of L_1 and b is an adjunct element in L_1 which is comparable with y such that $b < a$, a contradiction to the choice of a . Hence $[a]$ in $G_{\{0\}}(L'_1)$ does not contain any adjunct element of L'_1 .

Similarly, $[a']$ in $G_{\{0\}}(L'_2)$ does not contain any adjunct element of L'_2 .

Next, for the equivalence class $[a]$ in $G_{\{0\}}(L'_1)$, we have $\psi([a]) = \phi|_X([a]) = \{\phi(y) | y \in [a]\}$. Hence $a' = \phi(a) \in \psi([a])$. Let $\psi(b) = a'$, for $b \in [a]$. Then $b \sim a$, hence a and b are comparable in L'_1 . This gives $\psi(a)$ and $\psi(b) = a'$ are comparable in L'_2 . Suppose $\psi(a) < a'$, then we get a contradiction to the minimality of a' . Similarly, we can not have $a' < \psi(a)$. Hence $\psi(a) = a'$ in this case also. Hence we are done. \square

We, now, conclude the paper by solving Isomorphism Problem for the class of lower dismantlable lattices.

Proof of Theorem 1.1. It is clear that, if the lattices are isomorphic, then the zero-divisor graphs are isomorphic. Conversely, suppose that the zero-divisor graphs of lower dismantlable lattices are isomorphic. By Theorem 2.9, there exists an isomorphism $\phi : G_{\{0\}}(L_1) \rightarrow G_{\{0\}}(L_2)$ such that a is an adjunct element of L_1 if and only if $\phi(a)$ is an adjunct element of L_2 . Now, by Theorem 2.10, the lattices L_1 and L_2 are isomorphic. \square

In view of Theorem 2.1 and Remark 2.2, we have the following corollary.

Corollary 2.11. *Let T_1 and T_2 be two rooted trees. Then $G(T_1) \cong G(T_2)$ if and only if $T_1 \cong T_2$.*

3. CONCLUDING REMARKS

The Isomorphism Problem is one of the central problems of the zero-divisor graphs of algebraic structures as well as of ordered structures. LaGrange [16] proved that zero-divisor graphs of Boolean rings are isomorphic if and only if the rings are isomorphic. This result was extended by Mohammadian [18] for reduced rings with certain conditions. Recently, Joshi and Khiste [9] proved that the zero-divisor graphs of Boolean posets (Boolean lattices) are isomorphic if and only if the Boolean posets (Boolean lattices) are isomorphic.

It is well known that a Boolean ring can be uniquely determined by a Boolean lattice. Hence Joshi and Khiste [9] extended the result of LaGrange [16] to more general structures, namely Boolean posets. It is easy to observe that every Boolean lattice is an *SSC* lattice. Joshi et al. [14] proved the Isomorphism Problem for *SSC*-meet semilattices. We feel that with minor modifications, this result is true for *SSC* posets also. Thus in nutshell, Joshi et al. [14] extended the result of Joshi and Khiste [9]. From Theorem 1.15, it is clear that a lower dismantlable lattice L is *SSC* if and only if the associated basic block of L is L itself. This essentially proves the Isomorphism Problem for the class of lower dismantlable lattices which are basic blocks of itself. This motivate us to prove the Isomorphism Problem for the class of lower dismantlable lattices. Note that neither the class of *SSC* lattices nor the class of lower dismantlable lattices are contained in each other.

Mohammadian [18] proved the Isomorphism Problem for reduced rings. If R is a reduced commutative ring with unity then the ideal lattice, the set of all ideals of R , is a 0-distributive lattice,

see Joshi and Sarode [11, Lemma 2.8]. Hence one may expect that the Isomorphism Problem may be true for 0-distributive lattices/posets. Hence, we raise the following problem.

Problem. Let \mathcal{L} be the class of 0-distributive lattices such that the set of nonzero zero-divisors of $L \in \mathcal{L}$ is $L \setminus \{0, 1\}$. Is Isomorphism Problem true for the class \mathcal{L} ?

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